Using `IDAnalyticBH`

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Abstract

The `IDAnalyticBH` thorn contains a number of initial data sets for black-hole evolutions which can be specified analytically as metric and extrinsic curvature components. The initial data which is included in this thorn include single Schwarzschild and Kerr black holes, and multiple black hole Misner and Brill-Lindquist solutions.

1 Background

The `IDAnalyticBH` thorn exists as a central location to place any initial dataset for black hole evolution that can be specified analytically in terms of the metric, g_{ab}, and extrinsic curvature, K_{ab}.

The thorn extends the `admbase::initial_data` parameter by adding the following datasets:

`schwarzschild` Schwarzschild, in isotropic coordinates;

`kerr` Kerr, in Boyer-Lindquist coordinates;

`misner_bh` 2 Misner black holes;

`multiple_misner_bh` $N \geq 2$ Misner black holes; (Alas this code is currently broken; see the level 1 `CCTK_VWarn()` warning in `src/Misner_points.c`, function `Misner_init()`.)

`bl_bh` Multiple Brill-Lindquist black holes.

Initial data for lapse and shift can also be specified in this thorn.

The Cactus grid-functions corresponding to the initial data are inherited from the thorn `CactusEinstein/ADMBase`, along with the conformal factor grid-function, `psi` from `CactusEinstein/StaticConformal`, and its derivatives which are optionally set based on the value of the parameters `admbase::metric_type` and `staticconformal::conformal_storage`.

The `IDAnalyticBH` has been written and augmented over an number of years by many Cactus authors. These include John Baker, Steve Brandt, Carsten Gundlach, Joan Masso, Ed Seidel, Jonathan Thornburg, and Paul Walker. The following sections describe each of the initial datasets and their associated parameters in turn.

2 Schwarzschild

The Schwarzschild metric corresponds to a single, static, black hole. If the Cactus metric is specified as a conformal metric (by setting `admbase::metric_type="yes"`), then the metric is set using isotropic coordinates [1]:

$$ds^2 = -\left(\frac{2r - M}{2r + M}\right)^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{1}$$

with the Schwarzschild mass given by the single free parameter M. Thus, the three metric and extrinsic curvature have the values:

$$\hat{g}_{ab} = \psi^4 \delta_{ab}, \tag{2}$$

$$\psi = (1 + \frac{M}{2r}), \tag{3}$$

$$K_{ab} = 0. \tag{4}$$

The mass is specified using the parameter `idanalyticbh::mass`. The black hole is assumed to reside at the origin of the grid, corresponding to the location $x = y = z = 0$.

If the `admbase::metric_type` parameter has been set to `static conformal`, then the metric grid-functions (`admbase::gxx`, ..., `admbase::gzz`) are given as δ_{ab}, and the conformal factor `staticconformal::psi` is set to the value specified above. The derivatives of the conformal factor (`staticconformal::psix`, etc.) are determined analytically.

In order to give the lapse an initial profile which corresponds to isotropic lapse of the 4-metric specified above, use the parameter

    admbase::initial_lapse = "schwarz"

This will cause the `admbase::alp` grid-function to be initialised to the value:

$$\alpha = \frac{2r - M}{2r + M}. \tag{5}$$

Note that the Schwarzschild data has the following non-standard behaviour in response to the `admbase::metric_type` parameter. If the *physical* metric is requested (ie. `metric_type` is set to `"physical"`) then a *different* form of the Schwarzschild metric is set: Schwarzschild coordinates are set instead of the isotropic coordinates:

$$g_{xx} = g_{yy} = g_{zz} = 1 + 2M/r. \tag{6}$$

In order to carry out an evolution of a single Schwarzschild black hole of mass $m = 1$, using an initial lapse of $\alpha = 1$, you could modify your parameter file as follows:

    ActiveThorns = "... ADMBase StaticConformal IDAnalyticBH ..."

    admbase::metric_type = "static conformal"

    admbase::initial_data = "schwarzschild"
    admbase::initial_lapse = "one"   # or "schwarz" for isotropic lapse

    idanalyticbh::mass = 1.0

3   Kerr

Kerr initial data for an isolated rotating black hole is specified using the "quasi-isotropic" coordinates [2]:

$$ds^2 = \psi^4(dr^2 + r^2(d\theta^2 + \chi^2 \sin^2 \theta d\phi^2)), \tag{7}$$

where

$$\psi^4 = -2\frac{a^2}{r^2}\cos\theta\sin\theta, \tag{8}$$

$$\chi^2 = p^2/\Sigma, \tag{9}$$

2

$$p^2 = a^2 + r_k{}^2 - aB_\phi, \tag{10}$$

$$r_k = r + M + \frac{M^2 - a^2}{4r}, \tag{11}$$

$$B_\phi = -2Mr_k a \sin^2\theta/\Sigma, \tag{12}$$

$$\Sigma = r_k{}^2 + a^2\cos^2\theta. \tag{13}$$

The two free parameters are the Kerr mass, M, and angular momentum, a (assumed to be aligned with the z-axis). These are specified using the parameters `idanalyticbh::mass` and `idanalyticbh::a_kerr` respectively. *(Note that the default values for these parameters are $M = 2$ and $a = 0.1$.)* The black hole is assumed to reside at the centre of the coordinate system, at $x = y = z = 0$.

The `admbase::metric_type` parameter can be used to specify whether the metric should be conformal or not. If the metric is conformal, then ψ is initialised as a separate grid function, and it's first and second derivatives are calculated analytically and also stored as grid functions. Otherwise, the conformal factor is multiplied through in the expression for the 3-metric before the values of the `admbase::metric` variables are set. The extrinsic curvature is also determined analytically.

The gauge can be set to the Kerr lapse and shift with the parameters

```
idanalyticbh::initial_lapse = "kerr"
idanalyticbh::initial_shift = "kerr"
```

in which case the formulas

$$\alpha = \sqrt{\frac{\Delta}{p^2}}, \tag{14}$$

$$\beta^\phi = -2mr_k a/p^2, \tag{15}$$

where

$$\sqrt{\Delta} = r - \frac{m^2 - a^2}{4r}. \tag{16}$$

A set of parameters which initialise an evolution to use the Kerr intial data with mass $M = 1$ and angular momentum $a = 0.3$ are:

```
ActiveThorns = "... ADMBase StaticConformal IDAnalyticBH ..."

admbase::metric_type = "static conformal"

admbase::initial_data = "kerr"
admbase::initial_lapse = "kerr"
admbase::initial_shift = "kerr"

idanalyticbh::mass = 1.0
idanalyticbh::a_kerr = 0.3
```

4   Misner

The earliest suggestion for initial data that might be said to corresponding to multiple black holes was given by Misner in 1960 [3]. He provided a prescription for writing a metric connecting a pair of massive bodies, instaneously at rest, whose throats are connected by a wormhole. Using the method of images, this solution was generalised to describe any number of black holes whose throats connect two identical asymptotically flat spacetimes [4].

Two implementations of the Misner data are available. The first of these, "`misner_bh`", is due to Joan Masso, Ed Seidel and Karen Camarda, and implements the original two-throat solution. The more general solution was implemented by Steve Brandt and Carsten Gundlach, and is available as "`multiple_misner_bh`".
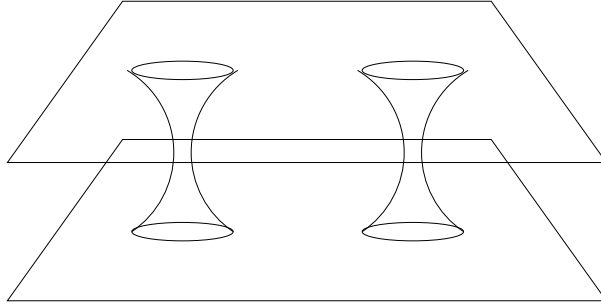
Figure 1: The topology of the Misner spacetime is that of a pair of asymptotically flat sheets connected by a number of Einstein-Rosen bridges. By construction, an exact isometry exists between the upper and lower sheet across the throats. The parameter μ_0 is a measure of the distance of a loop in the surface, passing through one throat and out the other.

4.1 Two-throat Misner data

The `misner_bh` initial data generates a metric of the form

$$ds^2 = -dt^2 + \psi^4(dx^2 + dy^2 + dz^2), \tag{17}$$

where the conformal factor ψ is given by

$$\psi = \sum_{n=-N}^{N} \frac{1}{\sinh(\mu_0 n)} \frac{1}{\sqrt{x^2 + y^2 + (z + \coth(\mu_0 n))^2}}. \tag{18}$$

The extrinsic curvature for the Misner data is zero.

The parameter μ_0 is a measure of the ratio of mass to separation of the throats, and is set using the parameter `idanalyticbh::mu`. For values less than $\mu \simeq 1.8$, the throats will have a single event horizon.

The summation limit N can be set using the parameter `idanalyticbh::nmax`. Ideally, it should tend to infinity, but in practice the default value of $N = 30$ works well enough for the applications that have been tested. The `misner_nbh` parameter is only used for the `multiple_misner_bh` multi-throat data, and will be ignored for the `misner_bh` initial data, which assumes two throats.

For the given metric, the ADM mass of the system is determined via

$$m = 4 \sum_{n=1}^{N} \frac{1}{\sinh(\mu_0 n)}. \tag{19}$$

This quantity is determined automatically and written to standard output.

If the conformal form of the metric is used (via the `admbase::metric_type` parameter), then derivatives of the conformal factor are computed analytically from the derivatives of the above expression for ψ.

To make use of the two black hole initial data, a variation of the following set of parameters can be used:

```
ActiveThorns = "... ADMBase StaticConformal IDAnalyticBH ..."

admbase::metric_type = "static conformal"

admbase::initial_data = "misner_bh"
idanalyticbh::mu = 2.2
```

5   Multiple-throat Misner data

The generalisation of the above form of data to multiple black holes is available as the `multiple_misner_bh` initial data set. The conformal factor is determined by recursively applying a Misner isometry condition to each of the black holes relative to the others.
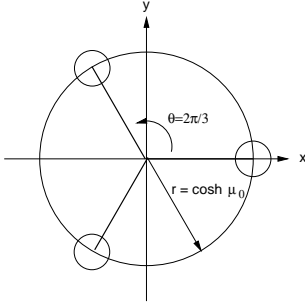
Figure 2: Configuration for three Misner throats using the `multiple_misner_bh` initial data.
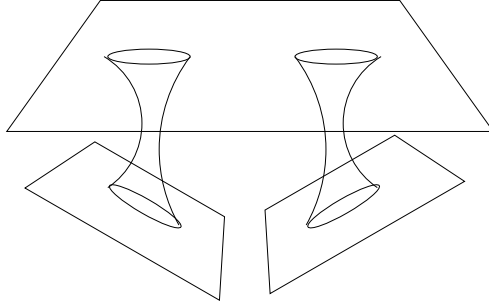

Figure 3: Two Brill-Lindquist throats connecting separate asymptotically flat regions.

The black holes are arranged at equal-spaced angles on a circle around the origin in the xy-plane. The radius of the circle is $\coth \mu_0$, where μ_0 is given by the `idanalyticbh::mu` parameter, and the first black hole lies on the x-axis (as in Figure 5).

The number of throats is given by the parameter `idanalyticbh::misner_nbh`, which defaults to 1 and has a hard-coded upper limit of 10. The number of terms used in the Misner expansion is controlled by the parameter `idanalyticbh::nmax`, which has a default value of 30.

For this version of the Misner data, derivatives of the conformal factor ψ are determined numerically by finite differencing, using values of ψ calculated at small distances from the point at which the derivative is to be evaluated. The size of the numerical stencil is hardcoded at $dx = 10^-6$.

As an example, a parameter file implementing 3 Misner black holes on a circle of radius $\cosh 4$ would use the following parameters:

```
ActiveThorns = "... ADMBase StaticConformal IDAnalyticBH ..."

admbase::metric_type = "static conformal"

admbase::initial_data = "multiple_misner_bh"

idanalyticbh::misner_nbh = 3
idanalyticbh::mu = 4
```

6   Brill-Lindquist

The Brill-Lindquist initial data is an alternate form of multi-throat data which differs from the Misner data mainly in its choice of spacetime topology. Whereas the Misner data presumes that the throats connect a pair of asymptotically flat spacetimes which are identical to each other, the Brill-Lindquist data connects each throat to a separate asymptotically flat region [5]. The form of the conformal factor is:

$$\psi = 1 + \sum_{i=1}^{N} \frac{m_i}{2r_i}, \tag{20}$$

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where the m_i and r_i are the masses and positions of the i particles.

The parameter specifying the number of black holes is `idanalyticbh::bl_nbh`. A maximum of four black holes can be specified. The mass and (x, y, z) position of the first black hole is given by `bl_M_1`, `bl_x0_1`, `bl_y0_1`, `bl_z0_1`, with corresponding parameters for the second to fourth black holes. Note that the default values for each of the position coordinates are 0.0, so that only the coordinates off of the axes must be specified.

If the conformal metric is used, then derivatives of the conformal factor are calculated from the analytic derivatives of the above expression for the conformal factor.

To initialise a run with a pair of Brill-Lindquist black holes with masses 1 and 2 and located at ± 1 on the y-axis, a set of parameters such as the following could be used:

```
ActiveThorns = "... ADMBase StaticConformal IDAnalyticBH ..."

admbase::metric_type = "static conformal"

admbase::initial_data = "bl_bh"

idanalyticbh::bl_nbh = 2

idanalyticbh::bl_M_1 = 1.0
idanalyticbh::bl_y0_1 = 1.0

idanalyticbh::bl_M_2 = 2.0
idanalyticbh::bl_y0_2 = -1.0
```

References

[1] See, for instance, p. 840 of: Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973) *Gravitation*, W. H. Freeman, San Francisco.

[2] Brandt, Steven R. and Seidel, Edward (1996) *Evolution of distorted rotating black holes. III. Initial data*, Phys. Rev., **D54**, 1403–1416.

[3] Misner, Charles W. (1960) *Wormhole Initial Conditions*, Phys. Rev., **118**, 1110–1111.

[4] Misner, Charles W. (1963) *The Method of Images in Geometrostatics*, Ann. Phys., **24**, 102–117.

[5] Brill, Dieter R., and Lindquist, Richard W. (1963) *Interaction Energy in Geometrostatics* Phys. Rev., **131**, 471–476.