Extracting Gravitational Waves and Other Quantities from Numerical Spacetimes

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Abstract

1 Introduction

Thorn Extract calculates first order gauge invariant waveforms from a numerical spacetime, under the basic assumption that, at the spheres of extract the spacetime is approximately Schwarzschild. In addition, other quantities such as mass, angular momentum and spin can be determined.

This thorn should not be used blindly, it will always return some waveform, however it is up to the user to determine whether this is the appropriate expected first order gauge invariant waveform.

2 Physical System

2.1 Wave Forms

Assume a spacetime $g_{\alpha\beta}$ which can be written as a Schwarzschild background $g^{Schwarz}_{\alpha\beta}$ with perturbations $h_{\alpha\beta}$:

$$g_{\alpha\beta} = g^{Schwarz}_{\alpha\beta} + h_{\alpha\beta} \tag{1}$$

with

$$\{g^{Schwarz}_{\alpha\beta}\}(t,r,\theta,\phi) = \begin{pmatrix} -S & 0 & 0 & 0 \\ 0 & S^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad S(r) = 1 - \frac{2M}{r} \tag{2}$$

The 3-metric perturbations γ_{ij} can be decomposed using tensor harmonics into $\gamma^{lm}_{ij}(t,r)$ where

$$\gamma_{ij}(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \gamma^{lm}_{ij}(t,r)$$

and

$$\gamma_{ij}(t,r,\theta,\phi) = \sum_{k=0}^{6} p_k(t,r)\mathbf{V}_k(\theta,\phi)$$

where $\{\mathbf{V}_k\}$ is some basis for tensors on a 2-sphere in 3-D Euclidean space. Working with the Regge-Wheeler basis (see Section 6) the 3-metric is then expanded in terms of the (six) standard Regge-Wheeler functions $\{c_1^{\times lm}, c_2^{\times lm}, h_1^{+lm}, H_2^{+lm}, K^{+lm}, G^{+lm}\}$ [19], [16]. Where each of the functions is either *odd* ($\times$) or *even* ($+$) parity. The decomposition is then written

$$\begin{aligned} \gamma^{lm}_{ij} &= c_1^{\times lm}(\hat{e}_1)^{lm}_{ij} + c_2^{\times lm}(\hat{e}_2)^{lm}_{ij} \\ &+ h_1^{+lm}(\hat{f}_1)^{lm}_{ij} + A^2 H_2^{+lm}(\hat{f}_2)^{lm}_{ij} + R^2 K^{+lm}(\hat{f}_3)^{lm}_{ij} + R^2 G^{+lm}(\hat{f}_4)^{lm}_{ij} \end{aligned} \tag{3}$$

which we can write in an expanded form as

$$\gamma^{lm}_{rr} = A^2 H_2^{+lm} Y_{lm} \tag{4}$$

$$
\begin{aligned}
\gamma^{lm}_{r\theta} &= -c_1^{\times lm}\frac{1}{\sin\theta}Y_{lm,\phi} + h_1^{+lm}Y_{lm,\theta} && (5)\\
\gamma^{lm}_{r\phi} &= c_1^{\times lm}\sin\theta Y_{lm,\theta} + h_1^{+lm}Y_{lm,\phi} && (6)\\
\gamma^{lm}_{\theta\theta} &= c_2^{\times lm}\frac{1}{\sin\theta}(Y_{lm,\theta\phi} - \cot\theta Y_{lm,\phi}) + R^2K^{+lm}Y_{lm} + R^2G^{+lm}Y_{lm,\theta\theta} && (7)\\
\gamma^{lm}_{\theta\phi} &= -c_2^{\times lm}\sin\theta\frac{1}{2}\left(Y_{lm,\theta\theta} - \cot\theta Y_{lm,\theta} - \frac{1}{\sin^2\theta}Y_{lm}\right) + R^2G^{+lm}(Y_{lm,\theta\phi} - \cot\theta Y_{lm,\phi}) && (8)\\
\gamma^{lm}_{\phi\phi} &= -\sin\theta c_2^{\times lm}(Y_{lm,\theta\phi} - \cot\theta Y_{lm,\phi}) + R^2K^{+lm}\sin^2\theta Y_{lm} + R^2G^{+lm}(Y_{lm,\phi\phi} + \sin\theta\cos\theta Y_{lm,\theta}) && (9)
\end{aligned}
$$

A similar decomposition allows the four gauge components of the 4-metric to be written in terms of *three* even-parity variables $\{H_0, H_1, h_0\}$ and the *one* odd-parity variable $\{c_0\}$

$$
\begin{aligned}
g^{lm}_{tt} &= N^2H_0^{+lm}Y_{lm} && (10)\\
g^{lm}_{tr} &= H_1^{+lm}Y_{lm} && (11)\\
g^{lm}_{t\theta} &= h_0^{+lm}Y_{lm,\theta} - c_0^{\times lm}\frac{1}{\sin\theta}Y_{lm,\phi} && (12)\\
g^{lm}_{t\phi} &= h_0^{+lm}Y_{lm,\phi} + c_0^{\times lm}\sin\theta Y_{lm,\theta} && (13)
\end{aligned}
$$

Also from $g_{tt} = -\alpha^2 + \beta_i\beta^i$ we have

$$\alpha^{lm} = -\frac{1}{2}NH_0^{+lm}Y_{lm} \tag{14}$$

It is useful to also write this with the perturbation split into even and odd parity parts:

$$g_{\alpha\beta} = g^{background}_{\alpha\beta} + \sum_{l,m} g^{lm,odd}_{\alpha\beta} + \sum_{l,m} g^{lm,even}_{\alpha\beta}$$

where (dropping some superscripts)

$$
\begin{aligned}
\{g^{odd}_{\alpha\beta}\} &= \begin{pmatrix}
0 & 0 & -c_0\frac{1}{\sin\theta}Y_{lm,\phi} & c_0\sin\theta Y_{lm,\theta}\\
. & 0 & -c_1\frac{1}{\sin\theta}Y_{lm,\phi} & c_1\sin\theta Y_{lm,\theta}\\
. & . & c_2\frac{1}{\sin\theta}(Y_{lm,\theta\phi} - \cot\theta Y_{lm,\phi}) & c_2\frac{1}{2}\left(\frac{1}{\sin\theta}Y_{lm,\phi\phi} + \cos\theta Y_{lm,\theta} - \sin\theta Y_{lm,\theta\theta}\right)\\
. & . & . & c_2(-\sin\theta Y_{lm,\theta\phi} + \cos\theta Y_{lm,\phi})
\end{pmatrix}\\
\{g^{even}_{\alpha\beta}\} &= \begin{pmatrix}
N^2H_0Y_{lm} & H_1Y_{lm} & h_0Y_{lm,\theta} & h_0Y_{lm,\phi}\\
. & A^2H_2Y_{lm} & h_1Y_{lm,\theta} & h_1Y_{lm,\phi}\\
. & . & R^2KY_{lm} + r^2GY_{lm,\theta\theta} & R^2(Y_{lm,\theta\phi} - \cot\theta Y_{lm,\phi})\\
. & . & . & R^2K\sin^2\theta Y_{lm} + R^2G(Y_{lm,\phi\phi} + \sin\theta\cos\theta Y_{lm,\theta})
\end{pmatrix}
\end{aligned}
$$

Now, for such a Schwarzschild background we can define two (and only two) unconstrained gauge invariant quantities $Q^\times_{lm} = Q^\times_{lm}(c_1^{\times lm}, c_2^{\times lm})$ and $Q^+_{lm} = Q^+_{lm}(K^{+lm}, G^{+lm}, H_2^{+lm}, h_1^{+lm})$, which from [3] are

$$
\begin{aligned}
Q^\times_{lm} &= \sqrt{\frac{2(l+2)!}{(l-2)!}}\left[c_1^{\times lm} + \frac{1}{2}\left(\partial_r c_2^{\times lm} - \frac{2}{r}c_2^{\times lm}\right)\right]\frac{S}{r} && (15)\\
Q^+_{lm} &= \frac{1}{\Lambda}\sqrt{\frac{2(l-1)(l+2)}{l(l+1)}}(4rS^2k_2 + l(l+1)rk_1) && (16)\\
&\equiv \frac{1}{\Lambda}\sqrt{\frac{2(l-1)(l+2)}{l(l+1)}}\left(l(l+1)S(r^2\partial_r G^{+lm} - 2h_1^{+lm}) + 2rS(H_2^{+lm} - r\partial_r K^{+lm}) + \Lambda rK^{+lm}\right) && (17)
\end{aligned}
$$

where

$$
\begin{aligned}
k_1 &= K^{+lm} + \frac{S}{r}(r^2\partial_r G^{+lm} - 2h_1^{+lm}) && (18)\\
k_2 &= \frac{1}{2S}\left[H_2^{+lm} - r\partial_r k_1 - \left(1 - \frac{M}{rS}\right)k_1 + S^{1/2}\partial_r(r^2S^{1/2}\partial_r G^{+lm} - 2S^{1/2}h_1^{+lm})\right] && (19)\\
&\equiv \frac{1}{2S}\left[H_2 - rK_{,r} - \frac{r-3M}{r-2M}K\right] && (20)
\end{aligned}
$$

NOTE: These quantities compare with those in Moncrief [16] by

$$\begin{aligned} \text{Moncriefs odd parity Q:} \qquad Q^{\times}_{lm} &= \sqrt{\frac{2(l+2)!}{(l-2)!}}Q \\ \text{Moncriefs even parity Q:} \qquad Q^{+}_{lm} &= \sqrt{\frac{2(l-1)(l+2)}{l(l+1)}}Q \end{aligned}$$

Note that these quantities only depend on the purely spatial Regge-Wheeler functions, and not the gauge parts. (In the Regge-Wheeler and Zerilli gauges, these are just respectively (up to a rescaling) the Regge-Wheeler and Zerilli functions). These quantities satisfy the wave equations

$$(\partial_t^2 - \partial_{r^*}^2)Q^{\times}_{lm} + S\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]Q^{\times}_{lm} = 0$$
$$(\partial_t^2 - \partial_{r^*}^2)Q^{+}_{lm} + S\left[\frac{1}{\Lambda^2}\left(\frac{72M^3}{r^5} - \frac{12M}{r^3}(l-1)(l+2)\left(1-\frac{3M}{r}\right)\right) + \frac{l(l-1)(l+1)(l+2)}{r^2\Lambda}\right]Q^{+}_{lm} = 0$$

where

$$\begin{aligned} \Lambda &= (l-1)(l+2) + 6M/r \\ r^* &= r + 2M\ln(r/2M - 1) \end{aligned}$$

3 Numerical Implementation

The implementation assumes that the numerical solution, on a Cartesian grid, is approximately Schwarzshild on the spheres of constant $r = \sqrt{(x^2+y^2+z^2)}$ where the waveforms are extracted. The general procedure is then:

- Project the required metric components, and radial derivatives of metric components, onto spheres of constant coordinate radius (these spheres are chosen via parameters).
- Transform the metric components and there derivatives on the 2-spheres from Cartesian coordinates into a spherical coordinate system.
- Calculate the physical metric on these spheres if a conformal factor is being used.
- Calculate the transformation from the coordinate radius to an areal radius for each sphere.
- Calculate the S factor on each sphere. Combined with the areal radius This also produces an estimate of the mass.
- Calculate the six Regge-Wheeler variables, and required radial derivatives, on these spheres by integration of combinations of the metric components over each sphere.
- Contruct the gauge invariant quantities from these Regge-Wheeler variables.

3.1 Project onto Spheres of Constant Radius

This is performed by interpolating the metric components, and if needed the conformal factor, onto the spheres. Although 2-spheres are hardcoded, the source code could easily be changed here to project onto e.g. 2-ellipsoids.

3.2 Calculate Radial Transformation

The areal coordinate $\hat{r}$ of each sphere is calculated by

$$\hat{r} = \hat{r}(r) = \left[\frac{1}{4\pi}\int\sqrt{\gamma_{\theta\theta}\gamma_{\phi\phi}}d\theta d\phi\right]^{1/2} \tag{21}$$

from which

$$\frac{d\hat{r}}{d\eta} = \frac{1}{16\pi\hat{r}} \int \frac{\gamma_{\theta\theta,\eta}\gamma_{\phi\phi} + \gamma_{\theta\theta}\gamma_{\phi\phi,\eta}}{\sqrt{\gamma_{\theta\theta}\gamma_{\phi\phi}}} \, d\theta d\phi \tag{22}$$

Note that this is not the only way to combine metric components to get the areal radius, but this one was used because it gave better values for extracting close to the event horizon for perturbations of black holes.

3.3 Calculate S factor and Mass Estimate

$$S(\hat{r}) = \left(\frac{\partial\hat{r}}{\partial r}\right)^2 \int \gamma_{rr} \, d\theta d\phi \tag{23}$$

$$M(\hat{r}) = \hat{r}\frac{1-S}{2} \tag{24}$$

3.4 Calculate Regge-Wheeler Variables

$$\begin{aligned}
c_1^{\times lm} &= \frac{1}{l(l+1)} \int \frac{\gamma_{\hat{r}\phi} Y^*_{lm,\theta} - \gamma_{\hat{r}\theta} Y^*_{lm,\phi}}{\sin\theta} d\Omega \\
c_2^{\times lm} &= -\frac{2}{l(l+1)(l-1)(l+2)} \int \left\{ \left(-\frac{1}{\sin^2\theta}\gamma_{\theta\theta} + \frac{1}{\sin^4\theta}\gamma_{\phi\phi} \right) (\sin\theta Y^*_{lm,\theta\phi} - \cos\theta Y^*_{lm,\phi}) \right. \\
&\quad \left. + \frac{1}{\sin\theta}\gamma_{\theta\phi}(Y^*_{lm,\theta\theta} - \cot\theta Y^*_{lm,\theta} - \frac{1}{\sin^2\theta} Y^*_{lm,\phi\phi}) \right\} d\Omega \\
h_1^{+lm} &= \frac{1}{l(l+1)} \int \left\{ \gamma_{\hat{r}\theta} Y^*_{lm,\theta} + \frac{1}{\sin^2\theta}\gamma_{\hat{r}\phi} Y^*_{lm,\phi} \right\} d\Omega \\
H_2^{+lm} &= S \int \gamma_{\hat{r}\hat{r}} Y^*_{lm} d\Omega \\
K^{+lm} &= \frac{1}{2\hat{r}^2} \int \left(\gamma_{\theta\theta} + \frac{1}{\sin^2\theta}\gamma_{\phi\phi} \right) Y^*_{lm} d\Omega \\
&\quad + \frac{1}{2\hat{r}^2(l-1)(l+2)} \int \left\{ \left(\gamma_{\theta\theta} - \frac{\gamma_{\phi\phi}}{\sin^2\theta} \right) \left(Y^*_{lm,\theta\theta} - \cot\theta Y^*_{lm,\theta} - \frac{1}{\sin^2\theta} Y^*_{lm,\phi\phi} \right) \right. \\
&\quad \left. + \frac{4}{\sin^2\theta}\gamma_{\theta\phi}(Y^*_{lm,\theta\phi} - \cot\theta Y^*_{lm,\phi}) \right\} d\Omega \\
G^{+lm} &= \frac{1}{\hat{r}^2 l(l+1)(l-1)(l+2)} \int \left\{ \left(\gamma_{\theta\theta} - \frac{\gamma_{\phi\phi}}{\sin^2\theta} \right) \left(Y^*_{lm,\theta\theta} - \cot\theta Y^*_{lm,\theta} - \frac{1}{\sin^2\theta} Y^*_{lm,\phi\phi} \right) \right. \\
&\quad \left. + \frac{4}{\sin^2\theta}\gamma_{\theta\phi}(Y^*_{lm,\theta\phi} - \cot\theta Y^*_{lm,\phi}) \right\} d\Omega
\end{aligned}$$

where

$$\gamma_{\hat{r}\hat{r}} = \frac{\partial r}{\partial\hat{r}}\frac{\partial r}{\partial\hat{r}}\gamma_{rr} \tag{25}$$

$$\gamma_{\hat{r}\theta} = \frac{\partial r}{\partial\hat{r}}\gamma_{r\theta} \tag{26}$$

$$\gamma_{\hat{r}\phi} = \frac{\partial r}{\partial\hat{r}}\gamma_{r\phi} \tag{27}$$

3.5 Calculate Gauge Invariant Quantities

$$Q^\times_{lm} = \sqrt{\frac{2(l+2)!}{(l-2)!}} \left[c_1^{\times lm} + \frac{1}{2}\left(\partial_{\hat{r}} c_2^{\times lm} - \frac{2}{\hat{r}} c_2^{\times lm} \right) \right] \frac{S}{\hat{r}} \tag{28}$$

$$Q^+_{lm} = \frac{1}{(l-1)(l+2) + 6M/\hat{r}} \sqrt{\frac{2(l-1)(l+2)}{l(l+1)}} (4\hat{r}S^2 k_2 + l(l+1)\hat{r}k_1) \tag{29}$$

where

$$k_1 = K^{+lm} + \frac{S}{\hat{r}}(\hat{r}^2 \partial_{\hat{r}} G^{+lm} - 2h_1^{+lm}) \tag{30}$$

$$k_2 = \frac{1}{2S}[H_2^{+lm} - \hat{r}\partial_{\hat{r}} k_1 - (1 - \frac{M}{\hat{r}S})k_1 + S^{1/2}\partial_{\hat{r}}(\hat{r}^2 S^{1/2}\partial_{\hat{r}} G^{+lm} - 2S^{1/2}h_1^{+lm} \tag{31}$$

4 Using This Thorn

Use this thorn very carefully. Check the validity of the waveforms by running tests with different resolutions, different outer boundary conditions, etc to check that the waveforms are consistent.

4.1 Basic Usage

4.2 Output Files

Although Extract is really an `ANALYSIS` thorn, at the moment it is scheduled at `POSTSTEP`, with the iterations at which output is performed determined by the parameter *itout*. Output files from `Extract` are always placed in the main output directory defined by `CactusBase/IOUtil`.

Output files are generated for each detector (2-sphere) used, and these detectors are identified in the name of each output file by `R1`, `R2`, ...

The extension denotes whether coordinate time (ṫl) or proper time (u̇l) is used for the first column.

- `rsch_R?.[tu]l`

  The extracted areal radius on each 2-sphere.

- `mass_R?.[tu]l`

  Mass estimate calculated from g_{rr} on each 2-sphere.

- `Qeven_R?_??.[tu]l`

  The even parity gauge invariate variable (*waveform*) on each 2-sphere. This is a complex quantity, the 2nd column is the real part, and the third column the imaginary part.

- `Qodd_R?_??.[tu]l`

  The odd parity gauge invariate variable (*waveform*) on each 2-sphere. This is a complex quantity, the 2nd column is the real part, and the third column the imaginary part.

- `ADMmass_R?.[tu]l`

  Estimate of ADM mass enclosed within each 2-sphere. (To produce this set `doADMmass = ‘‘yes’’`).

- `momentum_[xyz]_R?.[tu]l`

  Estimate of momentum at each 2-sphere. (To produce this set `do_momentum = ‘‘yes’’`).

- `spin_[xyz]_R?.[tu]l`

  Estimate of momentum at each 2-sphere. (To produce this set `do_spin = ‘‘yes’’`).

5 History

Much of the source code for Extract comes from a code written outside of Cactus for extracting waveforms from data generated by the NCSA G-Code for compare with linear evolutions of waveforms extracted from the Cauchy initial data. This work was carried out in collaboration with Karen Camarda and Ed Seidel.

6 Appendix: Regge-Wheeler Harmonics

$$
\begin{aligned}
(\hat{e}_1)^{lm} &= \begin{pmatrix} 0 & -\frac{1}{\sin\theta}Y_{lm,\phi} & \sin\theta Y_{lm,\theta} \\ . & 0 & 0 \\ . & 0 & 0 \end{pmatrix} \\
(\hat{e}_2)^{lm} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sin\theta}(Y_{lm,\theta\phi} - \cot\theta Y_{lm,\phi}) & . \\ 0 & -\frac{\sin\theta}{2}[Y_{lm,\theta\theta} - \cot\theta Y_{lm,\theta} - \frac{1}{\sin^2\theta}Y_{lm,\phi\phi}] & -\sin\theta[Y_{lm,\theta\phi} - \cot\theta Y_{lm,\phi}] \end{pmatrix} \\
(\hat{f}_1)^{lm} &= \begin{pmatrix} 0 & Y_{lm,\theta} & Y_{lm,\phi} \\ . & 0 & 0 \\ . & 0 & 0 \end{pmatrix} \\
(\hat{f}_2)^{lm} &= \begin{pmatrix} Y_{lm} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
(\hat{f}_3)^{lm} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_{lm} & 0 \\ 0 & 0 & \sin^2\theta Y_{lm} \end{pmatrix} \\
(\hat{f}_4)^{lm} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_{lm,\theta\theta} & . \\ 0 & Y_{lm,\theta\phi} - \cot\theta Y_{lm,\phi} & Y_{lm,\phi\phi} + \sin\theta\cos\theta Y_{lm,\theta} \end{pmatrix}
\end{aligned}
$$

7 Appendix: Transformation Between Cartesian and Spherical Coordinates

First, the transformations between metric components in (x, y, z) and (r, θ, ϕ) coordinates. Here, $\rho = \sqrt{x^2 + y^2} = r\sin\theta$,

$$
\begin{aligned}
\frac{\partial x}{\partial r} &= \sin\theta\cos\phi = \frac{x}{r} \\
\frac{\partial y}{\partial r} &= \sin\theta\sin\phi = \frac{y}{r} \\
\frac{\partial z}{\partial r} &= \cos\theta = \frac{z}{r} \\
\frac{\partial x}{\partial \theta} &= r\cos\theta\cos\phi = \frac{xz}{\rho} \\
\frac{\partial y}{\partial \theta} &= r\cos\theta\sin\phi = \frac{yz}{\rho} \\
\frac{\partial z}{\partial \theta} &= -r\sin\theta = -\rho \\
\frac{\partial x}{\partial \phi} &= -r\sin\theta\sin\phi = -y \\
\frac{\partial y}{\partial \phi} &= r\sin\theta\cos\phi = x \\
\frac{\partial z}{\partial \phi} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{rr} &= \frac{1}{r^2}(x^2\gamma_{xx} + y^2\gamma_{yy} + z^2\gamma_{zz} + 2xy\gamma_{xy} + 2xz\gamma_{xz} + 2yz\gamma_{yz}) \\
\gamma_{r\theta} &= \frac{1}{r\rho}(x^2z\gamma_{xx} + y^2z\gamma_{yy} - z\rho^2\gamma_{zz} + 2xyz\gamma_{xy} + x(z^2 - \rho^2)\gamma_{xz} + y(z^2 - \rho^2)\gamma_{yz}) \\
\gamma_{r\phi} &= \frac{1}{r}(-xy\gamma_{xx} + xy\gamma_{yy} + (x^2 - y^2)\gamma_{xy} - yz\gamma_{xz} + xz\gamma_{yz})
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\theta\theta} &= \frac{1}{\rho^2}(x^2 z^2 \gamma_{xx} + 2xyz^2\gamma_{xy} - 2xz\rho^2\gamma_{xz} + y^2 z^2 \gamma_{yy} - 2yz\rho^2\gamma_{yz} + \rho^4\gamma_{zz}) \\
\gamma_{\theta\phi} &= \frac{1}{\rho}(-xyz\gamma_{xx} + (x^2 - y^2)z\gamma_{xy} + \rho^2 y\gamma_{xz} + xyz\gamma_{yy} - \rho^2 x\gamma_{yz}) \\
\gamma_{\phi\phi} &= y^2\gamma_{xx} - 2xy\gamma_{xy} + x^2\gamma_{yy}
\end{aligned}
$$

or,

$$
\begin{aligned}
\gamma_{rr} &= \sin^2\theta\cos^2\phi\gamma_{xx} + \sin^2\theta\sin^2\phi\gamma_{yy} + \cos^2\theta\gamma_{zz} + 2\sin^2\theta\cos\phi\sin\phi\gamma_{xy} + 2\sin\theta\cos\theta\cos\phi\gamma_{xz} \\
&\quad +2\sin\theta\cos\theta\sin\phi\gamma_{yz} \\
\gamma_{r\theta} &= r(\sin\theta\cos\theta\cos^2\phi\gamma_{xx} + 2*\sin\theta\cos\theta\sin\phi\cos\phi\gamma_{xy} + (\cos^2\theta - \sin^2\theta)\cos\phi\gamma_{xz} + \sin\theta\cos\theta\sin^2\phi\gamma_{yy} \\
&\quad +(\cos^2\theta - \sin^2\theta)\sin\phi\gamma_{yz} - \sin\theta\cos\theta\gamma_{zz}) \\
\gamma_{r\phi} &= r\sin\theta(-\sin\theta\sin\phi\cos\phi\gamma_{xx} - \sin\theta(\sin^2\phi - \cos^2\phi)\gamma_{xy} - \cos\theta\sin\phi\gamma_{xz} + \sin\theta\sin\phi\cos\phi\gamma_{yy} \\
&\quad +\cos\theta\cos\phi\gamma_{yz}) \\
\gamma_{\theta\theta} &= r^2(\cos^2\theta\cos^2\phi\gamma_{xx} + 2\cos^2\theta\sin\phi\cos\phi\gamma_{xy} - 2\sin\theta\cos\theta\cos\phi\gamma_{xz} + \cos^2\theta\sin^2\phi\gamma_{yy} \\
&\quad -2\sin\theta\cos\theta\sin\phi\gamma_{yz} + \sin^2\theta\gamma_{zz}) \\
\gamma_{\theta\phi} &= r^2\sin\theta(-\cos\theta\sin\phi\cos\phi\gamma_{xx} - \cos\theta(\sin^2\phi - \cos^2\phi)\gamma_{xy} + \sin\theta\sin\phi\gamma_{xz} + \cos\theta\sin\phi\cos\phi\gamma_{yy} \\
&\quad -\sin\theta\cos\phi\gamma_{yz}) \\
\gamma_{\phi\phi} &= r^2\sin^2\theta(\sin^2\phi\gamma_{xx} - 2\sin\phi\cos\phi\gamma_{xy} + \cos^2\phi\gamma_{yy})
\end{aligned}
$$

We also need the transformation for the radial derivative of the metric components:

$$
\begin{aligned}
\gamma_{rr,\eta} &= \sin^2\theta\cos^2\phi\gamma_{xx,\eta} + \sin^2\theta\sin^2\phi\gamma_{yy,\eta} + \cos^2\theta\gamma_{zz,\eta} + 2\sin^2\theta\cos\phi\sin\phi\gamma_{xy,\eta} \\
&\quad +2\sin\theta\cos\theta\cos\phi\gamma_{xz,\eta} + 2\sin\theta\cos\theta\sin\phi\gamma_{yz,\eta} \\
\gamma_{r\theta,\eta} &= \frac{1}{r}\gamma_{r\theta} + r(\sin\theta\cos\theta\cos^2\phi\gamma_{xx,\eta} + \sin\theta\cos\theta\sin\phi\cos\phi\gamma_{xy,\eta} + (\cos^2\theta - \sin^2\theta)\cos\phi\gamma_{xz,\eta} \\
&\quad +\sin\theta\cos\theta\sin^2\phi\gamma_{yy,\eta} + (\cos^2\theta - \sin^2\theta)\sin\phi\gamma_{yz,\eta} - \sin\theta\cos\theta\gamma_{zz,\eta}) \\
\gamma_{r\phi,\eta} &= \frac{1}{r}\gamma_{r\phi} + r\sin\theta(-\sin\theta\sin\phi\cos\phi\gamma_{xx,\eta} - \sin\theta(\sin^2\phi - \cos^2\phi)\gamma_{xy,\eta} - \cos\theta\sin\phi\gamma_{xz,\eta} \\
&\quad +\sin\theta\sin\phi\cos\phi\gamma_{yy,\eta} + \cos\theta\cos\phi\gamma_{yz,\eta}) \\
\gamma_{\theta\theta,\eta} &= \frac{2}{r}\gamma_{\theta\theta} + r^2(\cos^2\theta\cos^2\phi\gamma_{xx,\eta} + 2\cos^2\theta\sin\phi\cos\phi\gamma_{xy,\eta} - 2\sin\theta\cos\theta\cos\phi\gamma_{xz,\eta} \\
&\quad +\cos^2\theta\sin^2\phi\gamma_{yy,\eta} - 2\sin\theta\cos\theta\sin\phi\gamma_{yz,\eta} + \sin^2\theta\gamma_{zz,\eta}) \\
\gamma_{\theta\phi,\eta} &= \frac{2}{r}\gamma_{\theta\phi} + r^2\sin\theta(-\cos\theta\sin\phi\cos\phi\gamma_{xx,\eta} - \cos\theta(\sin^2\phi - \cos^2\phi)\gamma_{xy,\eta} + \sin\theta\sin\phi\gamma_{xz,\eta} \\
&\quad +\cos\theta\sin\phi\cos\phi\gamma_{yy,\eta} - \sin\theta\cos\phi\gamma_{yz,\eta}) \\
\gamma_{\phi\phi,\eta} &= \frac{2}{r}\gamma_{\phi\phi} + r^2\sin^2\theta(\sin^2\phi\gamma_{xx,\eta} - 2\sin\phi\cos\phi\gamma_{xy,\eta} + \cos^2\phi\gamma_{yy,\eta})
\end{aligned}
$$

8 Appendix: Integrations Over the 2-Spheres

This is done by using Simpson's rule twice. Once in each coordinate direction. Simpson's rule is

$$\int_{x_1}^{x_2} f(x)dx = \frac{h}{3}[f_1 + 4f_2 + 2f_3 + 4f_4 + \ldots + 2f_{N-2} + 4f_{N-1} + f_N] + O(1/N^4) \tag{32}$$

N must be an odd number.

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